On the uniqueness of Sasaki-Einstein metrics

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Abstract

Let S be a compact Sasakian manifold which does not admit nontrivial Hamiltonian holomorphic vector fields. If there exists an Einstein-Sasakian metric on S, then it is unique.

1 Introduction

The aim of this paper is to show the uniquness theorem of positive Sasaki-Einstein metrics. An Sasaki-Einstein manifold admits a one dimensional Reeb foliation with a transversal Kähler-Einstein metric, which is studied from many view points between geometry and mathematical physics. Boyer, Galichi and Kollár obtained Sasaki-Einstein metrics on a family of the links of hypersurfaces of Brieskorn-Pham type, which includes exisotic spheres. Guantlett, Martelli, Sparks and Waldram discoverd that there exist irregular toric Einstein-Sasaki metrics which are not obtained as total spaces of orbibundles on Einstein-Kähler orbifolds ([7], [8]). These toric examples are much explored and Futaki, Ono and Wang showed that every toric positive Sasakian manifold admits Sasaki-Einstein metrics([6]). On a compact Kähler manifold with positive first Chern class, Bando and Mabuchi proved the uniqueness theorem of Kähler-Einstein metrics ([1]). K. Cho, A. Futaki and H. Ono proved that the toric Einstein-Sasaki metric is unique up to the automorphism of a toric Sasakian manifold ([4]). In the present paper, we show the following theorem,

Theorem 1.1. *Let (S, ξ, η, Φ) be a compact Sasakian manifold. We assume that S doesn't admit nontrivial Hamiltonian holomorphic vector fields. If S has a Sasaki-Einstein metric, then the Sasaki-Einstein metric is unique. In other words, if there are two Sasaki-Einstein metrics ω_1 and ω_2 on S, then $\omega_1 = \omega_2$.*

Our method is a generalization of Bando-Mabuchi's argument to Sasakian geometry. We construct functionals L, M, I and J on the space of Sasakian structures with basic first Chern class. These functionals satisfy the suitable properties as in Kähler geometry. The problem of Sasaki-Einstein metrics

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reduces to solving the following Monge-Ampère equation which gives rise to transversal Kähler-Einstein metrics with positive Ricci curvature,

$$\frac{(d\eta + \sqrt{-1}\partial_B \bar{\partial}_B u)^m}{(d\eta)^m} = \exp(-(2m+2)u + h)$$

The key point is to show the a priori estimate of C^0-norm of solutions u of the Monge-Ampère equation and an intriguing point is an estimate of infimum of u (lemma 5.8). The Monge-Ampère equation only gives the transversal Ricci curvature which does not lead a lower bound of the Ricci curvature by a positive constant. There is a difficulty of the C^0-estimate of u_t since we cannot apply the Myers theorem directly to obtain an estimate of the daimeter of S. We introduce a family of Sasakian structures $\{g_{u,\mu}\}$ whose contact forms are given by the multiplication of positive constant μ^{-1}. Under a suitable choice of μ, it follows that the Ricci curvature of $g_{\mu,\lambda}$ is bounded from below by a positive constant. Thus we can control their diameters by the Myeres theorem. An estimate of their volumes together with their diameters gives rise to the desired estimate of solutions u by using the estimate of the Green functions (see lemma 5.8 for more detail). Our method of the estimate is simple and effective in transversal Kähler metrics, which slightly different from the ordinary argument in Kähler geometry as transversal Kähler classes of the family $\{g_{u,\mu}\}$ are changing.

It must noted that Nitta obtained the theorem of uniqueness of Einstein -Sasakian metrics independently by the different method which heavily depends on several results in sub-Riemannian geometry such as the regularity of the space of piece-wise smooth horizontal paths ([11]).

An advantage of our method is that it is self-contained and could be generalized to more general transversal Kähler geometry which includes 3-Sasakian manifolds.

I would like to thank Ryushi Goto for helpful advice.

2 Sasakian manifold

In this section we give a brief explanation of Sasakian manifolds

Definition 2.1. *Let (S, g) be a Riemannian manifold of dimension $2m+1$ and $C(S)$ the cone $S \times \mathbb{R}_{>0}$ with $r \in \mathbb{R}_{>0}$. A Riemannian manifold (S, g) is said to be a Sasakian manifold if the cone manifold $(C(S), \bar{g}) = (S \times \mathbb{R}_{>0}, dr^2 + r^2 g)$ is a Kähler manifolds with complex structure J which satisfies*

$$\mathcal{L}_{r\frac{\partial}{\partial r}} J = 0,$$

where $\mathcal{L}_{r\frac{\partial}{\partial r}} J$ denotes the Lie derivative of J by the vector field $r\frac{\partial}{\partial r}$.

A Sasakian manifold S is often identified with the submanifold $\{r = 1\} = S \times \{1\} \subset C(S)$. Note that $C(S)$ is a real $2m$ dimensional manifold.

Definition 2.2. *We define a vector field ξ on S and a 1-form η on S by*

$$\xi = J\left(r\frac{\partial}{\partial r}\right), \qquad \eta(Y) = g(\xi, Y)$$

where Y is a smooth vector field on S. The vector field ξ is the Reeb field. We denote by $\mathcal{F}_\xi$ the 1-dimensional foliation generated by ξ which is called the Reeb foliation.

Then we see that

$$\eta(\xi) = 1, \qquad i_\xi d\eta = 0, \qquad (d\eta)^m \wedge \eta \neq 0. \tag{1}$$

The 1-form η is a contact form on S which defines a $2m$-dimensional subbundle D of the tangent bundle TS, where at each point $p \in S$ the fiber D_p of D is given by

$$D_p = \operatorname{Ker} \eta_p.$$

We call D the contact bundle. The contact bundle D gives the orthogonal decomposition of the tangent bundle TS

$$TS = D \oplus L_\xi$$

where L_ξ is the trivial bundle generated by the Reeb field ξ. A Sasakian manifold S is a foliated manifold with transversally Kähler structure. Then S admits foliated coordinates $\{U_\alpha\}$ compatible to the structure. The system of coordinates consists of an open covering $\{U_\alpha\}$ of S and a submersion $\pi_\alpha \colon U_\alpha \to V_\alpha \subset \mathbb{C}^m$ for each α such that

$$\pi_\alpha \circ \pi_\beta = \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta), \qquad U_\alpha \cap U_\beta \neq 0$$

is biholomorphic, where V_α is an open set of $\mathbb{C}^m$. On each V_α there is a Kähler structures given by the following. The restriction of the Sasaki metric g to D gives a well-defined Hermitian metric g_α^T on V_α under the canonical isomorphism

$$d\pi_\alpha \colon D_p \to T_{\pi_\alpha(p)} V_\alpha$$

for any $p \in U_\alpha$. Hence we have the transversally Hermitian structure on S. Let $(z^1, z^2, \ldots, z^m)$ be the local holomorphic coordinates on V_α. We pull back these to U_α and still write them as $(z^1, z^2, \ldots, z^m)$. Let x be the coordinate along the leaves with $\xi = \frac{\partial}{\partial x}$ Then $(x, z^1, z^2, \ldots, z^m)$ form local coordinates on U_α. We denote by $(D \otimes \mathbb{C})^{p,q}$ the set of forms of type (p, q) on S. Then $(D \otimes \mathbb{C})^{1,0}$ is spanned by the vectors of the form

$$\frac{\partial}{\partial z^i} - \eta\left(\frac{\partial}{\partial z^i}\right)\xi, \qquad i = 1, 2, \ldots, m.$$

Since $i_\xi d\eta = 0$,

$$d\eta\left(\frac{\partial}{\partial z^i} - \eta\left(\frac{\partial}{\partial z^i}\right)\xi, \overline{\frac{\partial}{\partial z^j} - \eta\left(\frac{\partial}{\partial z^j}\right)\xi}\right) = d\eta\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}\right).$$

Thus the fundamental 2-form ω_α of the Hermitian metric g^T_α on V_α is the same as the restriction of $d\eta$ to the slice $\{x = \text{constant}\}$ in U_α. Since the restriction of a closed 2-form to a submanifold is closed, then ω_α is closed. By this construction

$$\pi_\alpha \circ \pi_\beta^{-1} \colon \pi_\beta(U_\alpha \cap U_\beta) \to \pi_\alpha(U_\alpha \cap U_\beta)$$

gives an isometry of the Kähler structure.

Definition 2.3. *The collection of Kähler metrics $\{g^T_\alpha\}$ on $\{V_\alpha\}$ is called a transverse Kähler metric. Since they are isometric over the intersections we suppress α and denote it by g^T. We call coordinates system $(x, z^1, z^2, \ldots, z^m)$ given above a foliation chart.*

We also write Ric^T and s^T for Ricci curvature of g^T and scalar curvature of that. It should be emphasized that, though g^T are defined only locally on each V_α, the pull-back to U_α of the Kähler forms ω_α on V_α patch together and coincide with the global form $d\eta$ on S, and $d\eta$ can even be lifted to the cone $C(S)$ by pull-back. For this reason we often refer to $d\eta$ as the Kähler form of the transverse Kähler form of the transverse Kähler metric g^T. The next is a well known result.

Theorem 2.4 ([6])**.** *Let (S, g) be a Sasakian manifold. Then, we have*

$$\begin{aligned} &\mathrm{Ric}(X, \xi) = 2m\,\eta(X), \qquad \forall X \in TS \\ &\mathrm{Ric}(X, Y) = \mathrm{Ric}^T(X, Y) - 2g(X, Y), \qquad \forall X, Y \in D \end{aligned}$$

Definition 2.5. *A Sasakian manifold (S, g) is η-Einstein if there are two constants λ and ν such that*

$$\mathrm{Ric} = \lambda g + \nu\, \eta \otimes \eta.$$

Definition 2.6. *A Sasaki-Einstein manifold is a Sasakian manifold (S, g) with* $\mathrm{Ric} = 2mg$.

Definition 2.7. *A Sasakian manifold S is said to be transversely Kähler-Einstein Sasaki manifold if*

$$\mathrm{Ric}^T = \tau g^T$$

for some real constan τ.

It is well-known that if S is a transversely Kähler-Einstein Sasaki manifold if and only if (S, g) is η-Einstein (cf[2]). In fact, if $\mathrm{Ric}^T = \tau g^T$ then

$$\mathrm{Ric} = (\tau - 2)g + (2m + 2 - \tau)\eta \otimes \eta.$$

Conversely if $\mathrm{Ric} = \lambda g + \nu\eta \otimes \eta$ then

$$\mathrm{Ric}^T = (\lambda + 2)g^T.$$

3 Basic form

We introduce basic forms on Sasakian manifolds which is relevant to transversely Kähler-Einstein metrics on them. Let S be a compact Sasakian manifold of dimension $2m+1$.

Definition 3.1. *A p-form α on S is said to be basic if the following conditions hold*

$$i_\xi \alpha = 0, \qquad \mathcal{L}_\xi \alpha = 0.$$

Let Λ^p_B be the sheaf of germs of basic p-forms and Ω^p_B the set of all global sections of Λ^p_B.

It follows from (1) that $d\eta$ is a basic form. Let $(x, z^1, \dots, z^m)$ be the foliation chart on U_α as in definition 2.3. Then we write

$$\sum \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

for a form of type (p,q) on U_α. If $U_\alpha \cap U_\beta \neq \emptyset$ and $(y, w^1, \dots, w^m)$ is the foliation chart on U_β, then

$$\frac{\partial z^i}{\partial \bar{w}^j} = 0, \qquad \frac{\partial z^i}{\partial y} = 0.$$

Therefore, as in Kähler geometry, we have a notion of forms of type (p,q) which is independent of a choice of charts. If α is basic, then the coefficient $\alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}$ is a function which dose not depend on x. Thus we have a well-defined operators

$$\partial_B \colon \Lambda^{p,q}_B \to \Lambda^{p+1,q}_B$$
$$\bar{\partial}_B \colon \Lambda^{p,q}_B \to \Lambda^{p,q+1}_B.$$

It follows that $d\alpha$ is basic for a basic form α. Hence the exterior derivative d preserves the basic forms and we have the basic exterior derivative d_B and the complex of basic forms,

$$\cdots \to \Omega^p_B \to \Omega^{p+1}_B \to \cdots$$

which gives the basic cohomology group $H^p_B(S)$. We denote by $[\alpha]_B$ the basic cohomology class represented by a d_B-closed, basic p-form .

As in Kähler geometry, we have the decomposition $d_B = \partial_B + \bar{\partial}_B$. Let $d^c_B = \frac{\sqrt{-1}}{2}(\bar{\partial}_B - \partial_B)$. It is clear that

$$d_B d^c_B = \sqrt{-1}\partial_B \bar{\partial}_B, \qquad d^2_B = (d^c_B)^2 = 0$$

Let ∂^*_B be the adjoint operator of ∂_B and $\bar{\partial}^*_B$ the adjoint operator of $\bar{\partial}_B$ with respect to the transversally Kähler metric g^T. The basic Laplacian and the basic Dolbeault Laplacian are defined by

$$\triangle^B = d^*_B d_B + d_B d^*_B$$
$$\square^B = \bar{\partial}^*_B \bar{\partial}_B + \bar{\partial}_B \bar{\partial}^*_B.$$

On a Sasakian maifold, the $\partial\overline{\partial}$-lemma holds for basic forms.

Proposition 3.2 ([5]). *Let α and β be two basic forms of type $(1,1)$ on a compact Sasakian manifold S with $[\alpha]_B = [\beta]_B \in H^2_B(S)$ Then there is a basic function h such that*

$$\alpha = \beta + \sqrt{-1}\partial_B \bar{\partial}_B h.$$

As in [6] a new Sasakian structure fixing ξ and varying η is given by

$$\eta_\varphi = \eta + d^c_B \varphi$$

where φ is a small basic function preserving the positivity condition. Since $\dim_{\mathbb{R}} S = 2m+1$, a basic form with degree more than $2m+1$ is zero. Since $d\eta$ is basic, it follows from the Stokes theorem that we have

$$\int_S d_B \alpha \wedge \beta \wedge \eta = -(-1)^{\deg \alpha} \int_S \alpha \wedge d_B \beta \wedge \eta,$$

where α, β are basic and $\deg \alpha + \deg \beta = 2m-1$. Since $d^c_B \varphi$ is basic, for a basic $2m$-form γ, we also have

$$\int_S \gamma \wedge \eta = \int_S \gamma \wedge \eta_\varphi.$$

Therefore, in virtually, the results in Kähler geometry which can be proved only using the Stokes theorem, including the integration by parts still holds on compact Sasaki manifolds by using the contact form η.

Lemma 3.3.

$$\int_S (d\eta)^m \wedge \eta = \int_S (d\eta_\varphi)^m \wedge \eta_\varphi$$

Proof.

$$\int_S (d\eta_\varphi)^m \wedge \eta_\varphi = \int_S (d\eta + dd^c_B \varphi)^m \wedge (\eta + d^c_B \varphi) \tag{2}$$

$$= \int_S \sum \begin{pmatrix} m \\ k \end{pmatrix} (d\eta)^{m-k} \wedge (d_B d^c_B \varphi)^k \wedge (\eta + d^c_B \varphi) \tag{3}$$

$$= \int_S \sum \begin{pmatrix} m \\ k \end{pmatrix} (d\eta)^{m-k} \wedge (d_B d^c_B \varphi)^k \wedge \eta \tag{4}$$

$$+ \int_S \sum \begin{pmatrix} m \\ k \end{pmatrix} (d\eta)^{m-k} \wedge (d_B d^c_B \varphi)^k \wedge d^c_B \varphi. \tag{5}$$

The last term (5) is zero because it is a basic $(2m+1)$-form. In the case of $k \geq 1$, the term (4) is given by

$$d\left((d\eta)^{m-k} \wedge d^c_B \varphi \wedge (d_B d^c_B \varphi)^{k-1} \wedge \eta\right) \tag{6}$$

$$= (d\eta)^{m-k} \wedge (d_B d^c_B \varphi)^k \wedge \eta - (d\eta)^{m-k} \wedge d^c_B \varphi \wedge (d_B d^c_B \varphi)^{k-1} \wedge d\eta. \tag{7}$$

The second term of (7) is zero because it is a basic $(2m+1)$-form. Therefore the result follows from Stokes theorem

$$\int_S (d\eta)^m \wedge \eta = \int_S (d\eta_\varphi)^m \wedge \eta_\varphi.$$

□

Definition 3.4. *A collection of* $(1,1)$*-forms* ρ_α^T *on* $V_\alpha \subset \mathbb{C}^m$ *is given by*

$$\rho_\alpha^T = -\sqrt{-1}\partial\bar{\partial}\log\det(g_\alpha^T).$$

Then the collection of the pullback of forms $\pi_\alpha^*\rho_\alpha^T$ *defines a basic from of type* $(1,1)$ *which is called the transverse Ricci form. We denote by* Ric^T *the transverse Ricci form as well as the transverse Ricci tensor. To emphasize transverse Ricci form with respect to* $d\eta$*, we often write* $\mathrm{Ric}^T(d\eta)$*.*

There is a necessary condition for the existence of transversely Kähler-Einstein metrics.

Proposition 3.5 ([6])**.** *The transverse Ricci form* $\mathrm{Ric}^T(d\eta)$ *is represented by* $\tau d\eta$ *for some constant* τ *if and only if* $c_1(D)$ *is zero where* $D = \mathrm{Ker}\,\eta$ *is contact bundle.*

From now on we always assume that $c_1(D) = 0$.

4 Functionals on compact Sasakian manifolds

In this section, we define functionals on compact Sasakian manifolds which are analogous to the ones in Kähler geometry. We define Ω by

$$\Omega = \{\varphi \mid \varphi \text{ is basic and } d\eta_\varphi = \eta + d_B^c\varphi \text{ is positive definite}\}.$$

Thus η_φ gives a Sasakian structure for $\varphi \in \Omega$.

Proposition 4.1. *We assume that* $[\mathrm{Ric}^T(d\eta)]_B = (2m+2)[d\eta]_B$*. For every* $(\varphi, \varphi') \in \Omega \times \Omega$*, we define functionals* L*,* M *by*

$$\begin{aligned} L(\varphi,\varphi') &= \frac{1}{V}\int_a^b \left(\int_S \dot{\varphi}_t (d\eta_{\varphi_t})^m \wedge \eta_{\varphi_t}\right) dt \\ M(\varphi,\varphi') &= -\frac{1}{V}\int_a^b \left\{\int_S \dot{\varphi}_t (s^T(d\eta_{\varphi_t}) - m(2m+2))(d\eta_{\varphi_t})^m \wedge \eta_{\varphi_t}\right\} dt, \end{aligned}$$

where $V = \int_S (d\eta)^m \wedge \eta$ *and* $\{\varphi_t \mid a \le t \le b\}$ *is an arbitrary piecewise smooth path in* Ω *such that* $\varphi = \varphi_a$*,* $\varphi' = \varphi_b$*. Then* L, M *are independent of the choice of the path* $\{\varphi_t \mid a \le t \le b\}$*, therefore well-defined. Moreover,* L, M *satisfy the* 1*-cocycle condition, and for all* $C_1, C_2 \in \mathbb{R}$

$$\begin{aligned} L(\varphi, \varphi' + C_2) &= L(\varphi,\varphi') + C_2 \\ M(\varphi + C_1, \varphi' + C_2) &= M(\varphi,\varphi'). \end{aligned}$$

Proposition 4.2. *For every* $(\varphi,\varphi') \in \Omega \times \Omega$*, we define functionals* I*,* J *by*

$$\begin{aligned} I(\varphi,\varphi') &= \frac{1}{V}\int_S (\varphi' - \varphi)\left((d\eta_\varphi)^m - (d\eta_{\varphi'})^m\right) \wedge \eta \\ J(\varphi,\varphi') &= \frac{1}{V}\int_a^b \left(\int_S \dot{\varphi}_t \left((d\eta_\varphi)^m - (d\eta_{\varphi_t})^m\right) \wedge \eta\right) dt, \end{aligned}$$

where $V = \int_S (d\eta)^m \wedge \eta$ and $\{\varphi_t \mid a \leq t \leq b\}$ is an arbitrary piecewise smooth path in Ω such that $\varphi = \varphi_a$, $\varphi' = \varphi_b$. Then the following statements hold.

1. *$J(\varphi, \varphi') = -L(\varphi, \varphi') + \frac{1}{V}\int_S (\varphi' - \varphi)(d\eta_\varphi)^m \wedge \eta$, and J is independent of the choice of the path.*

2. *The functional J doesn't satisfy the 1-cocycle condition, but satisfy*
$$J(\varphi, \varphi') + J(\varphi', \varphi'') = J(\varphi, \varphi'') - \frac{1}{V}\left(\int_S (\varphi'' - \varphi')\left((d\eta_\varphi)^m - (d\eta_{\varphi'})^m\right) \wedge \eta\right).$$

3. *Let C be a constant, then*
$$\begin{aligned} I(\varphi, \varphi' + C) &= I(\varphi, \varphi') \\ J(\varphi, \varphi' + C) &= J(\varphi, \varphi'). \end{aligned}$$

4. *Let $\{\varphi_t\}$ be a family of basic functions, then*
$$\frac{d}{dt}\left(I(\varphi, \varphi_t) - J(\varphi, \varphi_t)\right) = \frac{1}{V}\int_S (\varphi_t - \varphi)\left(\Box^B_{\varphi_t}\frac{d}{dt}\varphi_t\right)(d\eta_{\varphi_t})^m \wedge \eta.$$

5. *$I, I - J, J$ are non-negative functionals on Ω, and we have*
$$0 \leq I(\varphi, \varphi') \leq (m+1)(I(\varphi, \varphi') - J(\varphi, \varphi')) \leq mI(\varphi, \varphi').$$

The propositions 4.1 and 4.2 can be proved by a similar method as in the Kähler cases (see [9]) by applying the procedure in the proof of the lemma 3.3 in the secion 3.

Definition 4.3. *A complex vector field X on a Sasakian manifold is called a Hamiltonian holomorphic vector field if*

1. *$d\pi_\alpha(X)$ is a holomorphic vector field on V_α.*

2. *The basic function $u_X := \sqrt{-1}\eta(X)$ satisfies*
$$\bar{\partial}_B u_X = -\frac{\sqrt{-1}}{2} i(X) d\eta.$$

   Such a function u_X is called a Hamiltonian function.

Let $(x, z^1, \ldots, z^m)$ be a foliation chart on U_α. Then we can write a Hamiltonian holomorphic vector field X as
$$X = \eta(X)\frac{\partial}{\partial x} + \sum_{i=1}^m X^i \frac{\partial}{\partial z^i} - \eta\left(\sum_{i=1}^m X^i \frac{\partial}{\partial z^i}\right)\frac{\partial}{\partial x},$$
where
$$\tilde{X} = X + \sqrt{-1}\left(\eta(X) - \eta\left(\sum_{i=1}^m X^i \frac{\partial}{\partial z^i}\right)\right) r\frac{\partial}{\partial r}$$

is a holomorphic vector field on $C(S)$ (see [6]).

Since $0 \in \Omega$, we abuse a notation as

$$M(d\eta_\varphi) = M(0, \varphi).$$

It is shown that φ is a critical point of M on Ω if and only if $d\eta_\varphi$ is a transversely Kähler-Einstein metric (see [6]).

5 Monge-Ampère equation

We assume that $[\mathrm{Ric}^T(d\eta)]_B = (2m+2)[d\eta]_B$.

Then it follows from the proposition 3.2 that there exists a function h such that

$$\begin{aligned} &\mathrm{Ric}^T(d\eta) - (2m+2)d\eta = \sqrt{-1}\partial_B\bar{\partial}_B h \\ &\int_S (e^h - 1)(d\eta)^m \wedge \eta = 0. \end{aligned}$$

As in Kähler geometry, the Ricci curvature of $d\eta_u = d\eta + \sqrt{-1}\partial_B\bar{\partial}_B u$ is given by

$$\begin{aligned} \mathrm{Ric}^T(d\eta_u) &= -\sqrt{-1}\partial_B\bar{\partial}_B \log(d\eta_u)^m \\ &= -\sqrt{-1}\partial_B\bar{\partial}_B \log\left(\frac{(d\eta_u)^m}{(d\eta)^m}\right) + \mathrm{Ric}^T(d\eta) \\ &= -\sqrt{-1}\partial_B\bar{\partial}_B \log\left(\frac{(d\eta_u)^m}{(d\eta)^m}\right) \\ &\qquad + \sqrt{-1}\partial_B\bar{\partial}_B(-(2m+2)u + h) + (2m+2)d\eta_u. \end{aligned}$$

Hence, $d\eta_u$ is a transversely Kähler-Einstein metric if and only if $d\eta_u$ satisfies the following equation

$$-\sqrt{-1}\partial_B\bar{\partial}_B \log\left(\frac{(d\eta_u)^m}{(d\eta)^m}\right) + \sqrt{-1}\partial_B\bar{\partial}_B(-(2m+2)u + h) = 0$$

which is equivalent to the Monge-Ampère equation,

$$\frac{(d\eta + \sqrt{-1}\partial_B\bar{\partial}_B u)^m}{(d\eta)^m} = \exp(-(2m+2)u + h)$$

In order to prove the uniqueness of solutions, we consider two families of equations parametrized by $t \in [0,1]$:

$$\frac{(d\eta + \sqrt{-1}\partial_B\bar{\partial}_B u)^m}{(d\eta)^m} = \exp(-t(2m+2)u + h) \tag{8}$$

$$\frac{(d\eta + \sqrt{-1}\partial_B\bar{\partial}_B u)^m}{(d\eta)^m} = \exp(-t(2m+2)u - (2m+2)L(0,u) + h). \tag{9}$$

For a solution u of (8), $u - \dfrac{1}{t+1}L(0,u)$ is a solution of (9). On the other hands, if $t > 0$, for a solution u of (9), $u + \dfrac{1}{t}L(0,u)$ is a solution of (8). Therefore (8) and (9) are same for $t \in (0,1]$, but a difference occurs to $t = 0$. If u is a solution of (8), then $u +$ (constant) is also a solution for $t = 0$, but not for $t > 0$. Sicne this is inconvenient to prove the uniqueness, we introduce the equation (9) for this problem. We set

$$\begin{aligned} I_1 =& \{t \in [0,1] \mid \text{ the equation (8) has solutions for } t\} \\ I_2 =& \{t \in [0,1] \mid \text{ the equation (9) has solutions for } t\}. \end{aligned}$$

If we prove I_1 is open and close, then there exists a solution for $t = 1$ and this solution gives a transversely Kähler-Einstein metric.

We remark that a solution u of (8) or (9) satisfies

$$\begin{aligned} \mathrm{Ric}^T(d\eta_u) &= t(2m+2)d\eta_u + (1-t)(2m+2)d\eta \\ \therefore \mathrm{Ric}^T(d\eta_u) &\geq t(2m+2)d\eta_u. \end{aligned}$$

From now on we always assume that $\mathrm{Ric}^T(d\eta) - (2m+2)d\eta = \sqrt{-1}\partial_B\bar{\partial}_B h$.

5.1 Openness

In this subsection, we shall prove that I_2 is open.

Definition 5.1. *A Hamiltonian holomorphic vector field X is called a normalized Hamiltonian holomorphic vector field if the Hamiltonian function u_X satisfies*

$$\int_S u_X e^h (d\eta)^m \wedge \eta = 0.$$

Proposition 5.2 (theorem 5.1 of [6])**.** *Let $\square_h^B$ be the Laplacian with respect to Hermitian metric* $\exp(h)d\eta$. *Then we have*

1. *The first eigenvalue of $\square_h^B$ is greater than or equal to $2m+2$.*
2. $\mathrm{Ker}(\square_h^B - (2m+2))$ *is isomorphic to* $\{X \mid$ *normalized Hamiltonian holomorphic vector fields*$\}$. *The correspondence is given by*

$$u \mapsto u\xi + \sum (g^T)^{i\bar{j}} \frac{\partial u}{\partial \bar{z}^j}\frac{\partial}{\partial z^i} + \eta\left(\sum (g^T)^{i\bar{j}} \frac{\partial u}{\partial \bar{z}^j}\frac{\partial}{\partial z^i}\right)\xi.$$

Let V be a open subset in $\mathbb{C}^m$ and $\exp(h)(d\eta)^m$ a Hermitian metric on the anti-canonical line bundle K_V^{-1}. We denote by $\square^{\bar{\partial}}_{K_V^{-1},h}$ the Laplacian with respect to this metirc . Let $R_{V,h}$ be the curvature of the canonical connection. Then we have

$$\sqrt{-1}R_{V,h} = (2m+2)d\eta.$$

By the Kodaira-Akitsuki-Nakano identity on $\Lambda^{m,1}(K_V^{-1})$, we have

$$\square^{\bar{\partial}}_{K_V^{-1},h} = \square^{\partial}_{K_V^{-1},h} + (2m+2).$$

Proposition 5.3. *The intervals I_1 and I_2 satisfy the followings,*

(i) $0 \in I_1,\ I_2$.

(ii) I_2 *is a open set in* $[0,1)$.

(iii) If S doesn't have non-trivial normalized Hamiltonian holomorphic vector fields, both I_1 and I_2 are open in a neighborhood of 1.

Proof. At first we shall show (i). The equation (8) admits a solution u for $t=0$ by [12] and [5]. Thus $0 \in I_1$. For a solution u of I_1 of $t=0$, $u - L(0,u)$ is a solution of I_2 of $t=0$.

Next we shall show (ii). We define Φ_1 for (8) by

$$\Phi_1 \colon \Omega \times I \to C_B^{0,\varepsilon}(S)$$

$$\Phi_1(u,t) = \log\left(\frac{(d\eta + \sqrt{-1}\partial_B\bar{\partial}_B u)^m}{(d\eta)^m}\right) + t(2m+2)u - h,$$

where $u \in C_B^{2,\varepsilon}(S)$. We also define Φ_2 for (9) by

$$\Phi_2 \colon \Omega \times I \to C_B^{0,\varepsilon}(S)$$

$$\begin{aligned}\Phi_2(u,t) =& \log\left(\frac{(d\eta + \sqrt{-1}\partial_B\bar{\partial}_B u)^m}{(d\eta)^m}\right)\\ &+ t(2m+2)u + (2m+2)L(0,u) - h.\end{aligned}$$

where $u \in C_B^{2,\varepsilon}(S)$. By differentiating Φ_1 and Φ_2 in the $\dot{u}$ direction at u for a fixed t , we have

$$\begin{aligned}(d\Phi_1)_u(\dot{u}) &= -\Box_u^B \dot{u} + t(2m+2)\dot{u}\\ (d\Phi_2)_u(\dot{u}) &= -\Box_u^B \dot{u} + t(2m+2)\dot{u} + \frac{2m+2}{V}\int_S \dot{u}(d\eta_u)^m \wedge \eta.\end{aligned}$$

Our discussion is divided into two cases : $t=0$ and $t \neq 0$.

1. In the case of $t=0$.

   Let u be a solution of (9) and $\dot{u} \in \mathrm{Ker}(d\Phi_2)_u$. Then we have

   $$\Box_u^B \dot{u} = \frac{2m+2}{V}\int_S \dot{u}(d\eta_u)^m \wedge \eta.$$

   When we integrate this in S, we have

   $$\begin{aligned}0 =& \int_S \Box_u^B \dot{u}(d\eta_u)^m \wedge \eta\\ =& \int_S \left(\frac{2m+2}{V}\int_S \dot{u}(d\eta_u)^m \wedge \eta\right)(d\eta_u)^m \wedge \eta\\ =&(2m+2)\int_S \dot{u}(d\eta_u)^m \wedge \eta.\end{aligned}$$

Therefore we have

$$\Box_u^B \dot{u} = \frac{2m+2}{V}\int_S \dot{u}(d\eta_u)^m \wedge \eta = 0.$$

Hence $\dot{u}$ is a constant. Since the integration is 0, the constant is 0. Therefore $\dot{u} = 0$. From the implicit function theorem, there exists an open neighborhood of 0 which is included in I_2.

2. In the case of $t \in (0,1]$.

   Since (9) is not different from (8) for $t \in (0,1]$, It suffices to prove that for I_1. Let u be a solution of (8) with $\dot{u} \in \mathrm{Ker}(d\Phi_1)_u$. Then we have

   $$\Box_u^B \dot{u} = t(2m+2)\dot{u}.$$

   We consider $\bar{\partial}_B \dot{u}$ to be an element of $\Lambda^{m,1}(K_V^{-1})$. Then by the Kodaira-Akitsuki-Nakano identity, we have

   $$\Box_u^B = \Box^{\bar{\partial}}_{K_V^{-1},u} = \Box^{\partial}_{K_V^{-1},u} + [\mathrm{Ric}^T(d\eta_u), \Lambda].$$

   Therefore we have

   $$\begin{aligned}\frac{1}{2m+2}t\|\bar{\partial}_B\dot{u}\|^2 =&(\Box_u^B\bar{\partial}_B\dot{u}, \bar{\partial}_B\dot{u})\\ =&(\Box^{\partial}_{K_V^{-1},u}\bar{\partial}\dot{u}, \bar{\partial}\dot{u}) + ([\mathrm{Ric}^T(d\eta_u), \Lambda]\bar{\partial}\dot{u}, \bar{\partial}\dot{u})\\ =&(\Box^{\partial}_{K_V^{-1},u}\bar{\partial}\dot{u}, \bar{\partial}\dot{u}) + (\mathrm{Ric}^T(d\eta_u)\Lambda\bar{\partial}\dot{u}, \bar{\partial}\dot{u}).\end{aligned}$$

   In the case of $0 < t < 1$, there exists a positive constant ε such that

   $$\mathrm{Ric}^T(d\eta_u) > (t+\varepsilon)(2m+2)d\eta_u.$$

   Since $(\Box^{\partial}_{K_V^{-1},u}\bar{\partial}\dot{u}, \bar{\partial}\dot{u}) \geq 0$, we have

   $$\begin{aligned}0 \leq& t(2m+2)\|\bar{\partial}\dot{u}\|^2 - (\mathrm{Ric}^T(d\eta_u)\Lambda\bar{\partial}\dot{u}, \bar{\partial}\dot{u})\\ <& t(2m+2)\|\bar{\partial}\dot{u}\|^2 - ((t+\varepsilon)(2m+2)L\Lambda\bar{\partial}\dot{u}, \bar{\partial}\dot{u})\\ =& t(2m+2)\|\bar{\partial}\dot{u}\|^2 - (t+\varepsilon)(2m+2)([L,\Lambda]\bar{\partial}\dot{u}, \bar{\partial}\dot{u})\\ =& t(2m+2)\|\bar{\partial}\dot{u}\|^2 - (t+\varepsilon)(2m+2)(\bar{\partial}\dot{u}, \bar{\partial}\dot{u})\\ =& -\varepsilon(2m+2)\|\bar{\partial}\dot{u}\|^2 < 0\end{aligned}$$

   Hence $\bar{\partial}\dot{u} = 0$. Thus $\dot{u}$ is a constant and we have

   $$0 = -\Box_u^B\dot{u} + t(2m+2)\dot{u} = t(2m+2)\dot{u}$$
   $$\therefore \dot{u} = 0.$$

   By the implicit function theorem, $I_1 \cap (0,1)$ is a open set. Therefore $I_2 \cap (0,1)$ is a open set.

Finally we shall show (iii). It also suffices to show that for I_1. In the case of $t = 1$, solutions are transversely Kähler-Einstein metrics. If S doesn't have non-trivial normalized Hamiltonian holomorphic vector fields, We have $\mathrm{Ker}(\Box_u^B - (2m+2)) = 0$ by proposition 5.2. Hence, by the implicit function theorem, I_1 is an open set in neighborhood of 1. Therefore I_2 is also an open set in neighborhood of 1. ☐

5.2 Estimates for closeness

In this subsection, we shall obtain estimates to prove that I_2 is close.

By Yau [12] and El-Kacimi [5], if there exists a C^0-estimate of solutions of the Monge-Ampère equation

$$\sup_S |u| \leq C,$$

then we obtain a $C^{2,\varepsilon}$-estimate,

$$\|u\|_{C^{2,\varepsilon}} \leq C'$$

where C' is a constant which dosen't depend on u. Afterward, it follows form the Ascoli-Arzelà theorem that I_2 is closed.

Lemma 5.4. *Let u_t be a C^∞-solution of* (9). *Then we have*

$$\frac{dM(0,u_t)}{dt} = -(2m+2)(1-t)\frac{d}{dt}(I(0,u_t) - J(0,u_t)) \leq 0.$$

Proof. Let $\eta_t = \eta + d_B^c u_t$. By the definition of h and (9), we have

$$\mathrm{Ric}^T(d\eta_t) = (2m+2)d\eta_t - \sqrt{-1}(2m+2)(1-t)\partial_B\bar{\partial}_B u_t$$

By taking the trace with respect to the transversal Kähler form $d\eta_t$, we have

$$s^T(d\eta_t) - m(2m+2) = (2m+2)(1-t)\Box_{u_t}^B u_t.$$

Hence we obtain

$$\begin{aligned}\frac{dM(0,u_t)}{dt} &= -\frac{1}{V}\int_S \dot{u}_t(s^T(d\eta_t) - m(2m+2))(d\eta_t)^m \wedge \eta \\ &= -\frac{1}{V}\int_S \dot{u}_t(2m+2)(1-t)\Box_{u_t}^B u_t (d\eta_t)^m \wedge \eta \\ &= -(2m+2)(1-t)\frac{1}{V}\int_S u_t \Box_{u_t}^B \dot{u}_t (d\eta_t)^m \wedge \eta.\end{aligned}$$

Therefore the first equation of this lemma is proved by proposition 4.2.

We take a logarithm of (9) and differentiate by t. Then we have

$$-\Box_{u_t}^B \dot{u}_t = -(2m+2)\left(u_t + t\dot{u}_t + \frac{1}{V}\int_S \dot{u}_t (d\eta_t)^m \wedge \eta\right).$$

Therefore we obtain

$$\begin{aligned}
&(2m+2)\frac{d}{dt}(I(0,u_t)-J(0,u_t))\\
=&\frac{(2m+2)}{V}\int_S \dot{u}_t\Box^B_{u_t}u_t(d\eta_t)^m\wedge\eta\\
=&\frac{1}{V}\int_S \dot{u}_t\Box^B_{u_t}\left(\Box^B_{u_t}\dot{u}_t - t(2m+2)\dot{u}_t - \frac{(2m+2)}{V}\int_S \dot{u}_t(d\eta_t)^m\wedge\eta\right)(d\eta_t)^m\wedge\eta\\
=&\frac{1}{V}\int_S \dot{u}_t\bar{\partial}^*_B\bar{\partial}_B\left(\bar{\partial}^*_B\bar{\partial}_B\dot{u}_t - t(2m+2)\dot{u}_t\right)(d\eta_t)^m\wedge\eta\\
=&\frac{1}{V}\int_S \left(\bar{\partial}_B\dot{u}_t, \Box^B_{u_t}\bar{\partial}_B\dot{u}_t - t(2m+2)\bar{\partial}_B\dot{u}_t\right)(d\eta_t)^m\wedge\eta.
\end{aligned}$$

Here we assume $\bar{\partial}_B\dot{u}_t$ as an element of $A^{m,1}(K_V^{-1})$ as in proposition 5.3. Then we have

$$\begin{aligned}
&\left(\bar{\partial}_B\dot{u}_t, \Box^B_{u_t}\bar{\partial}_B\dot{u}_t - t(2m+2)\bar{\partial}_B\dot{u}_t\right)\\
=&\left(\Box^{\partial}_{K_V^{-1},u_t}\bar{\partial}\dot{u}_t, \bar{\partial}\dot{u}_t\right) + \left(([\mathrm{Ric}^T(d\eta_{u_t}),\Lambda] - (2m+2)t)\bar{\partial}\dot{u}_t, \bar{\partial}\dot{u}_t\right)\\
\geq&\left(\Box^{\partial}_{K_V^{-1},u_t}\bar{\partial}\dot{u}_t, \bar{\partial}\dot{u}_t\right) + (2m+2)\left([tL,\Lambda] - t)\bar{\partial}\dot{u}_t, \bar{\partial}\dot{u}_t\right)\\
\geq&0.
\end{aligned}$$

Therefore we obtain

$$\frac{dM(0,u_t)}{dt} = -(2m+2)(1-t)\frac{d}{dt}(I(0,u_t)-J(0,u_t)) \leq 0.$$

$\square$

The following is a well known fact on the Green function on compact Riemannian manifolds

Fact 5.5 ([10]). *Let (S,g) be a compact Riemannian manifold of dimension $2m+1$. Then there exists a Green function $G(x,y)$ which satisfyies*

$$u(x) = \frac{1}{\mathrm{Vol}(S,g)}\int_S u(y)dV_g(y) + \int_S G(x,y)(\triangle u)(y)dV_g(y)$$

for all $u \in C^\infty(S)$ and

$$\int_S G(x,y)dV_g(y) = 0,$$

where $\triangle$ is the Lapacian and dV_g is the volume form. In addition, we assume

$$\mathrm{diam}(X,g)^2\,\mathrm{Ric}(g) \geq -(m-1)\varepsilon^2 g$$

for a constant $\varepsilon \geq 0$. Then there exists a constant $\gamma(m,\varepsilon)$ which depends on only m and ε and we have

$$G(x,y) \geq -\gamma(m,\varepsilon)\frac{\mathrm{diam}(S,g)^2}{\mathrm{Vol}(S,g)}$$

for the Green function of (S, g)*.*

We remark that a solution u of (8) or (9) satisfying

$$\mathrm{Ric}^T(d\eta_u) = t(2m+2)d\eta_u + (1-t)(2m+2)d\eta$$

Hence we have

$$\mathrm{Ric}^T(d\eta_u) \geq t(2m+2)d\eta_u.$$

We introduce a family of contact structures by the multiplication of positive constant μ,

$$\eta_{u,\mu} = \mu^{-1}\eta_u \tag{10}$$

$$\xi_\mu = \mu\xi \tag{11}$$

Then we see that $(\eta_{u,\mu}, \xi_\mu)$ gives a Sasakian structure with the metric $g_{u,\mu}$ on S. The transversal metric $g^T_{u,\mu}$ is given by $g^T_{u,\mu} = \mu^{-1}g^T_u$. The volume form of $g_{u,\mu}$ is given by

$$\eta_{u,\mu} \wedge (d\eta_{u,\mu})^m = \mu^{-(m+1)}\eta_u \wedge (d\eta_u)^m. \tag{12}$$

Let $\Box_{u,\mu}$ be the Laplacian with the Green operator $G_{u,\mu}$ and $\mathrm{Ric}_{u,\mu}$ the Ricci tensor with respect to $g_{u,\mu}$.

Proposition 5.6. *Let* (S, g) *be a compact Sasakian manifold and* u *a solution of (8) or (9). If we set* $\mu = t^{-1}$*, then we have estimates of the volume and the diameter with respect to the metric* $g_{u,\mu}$*,*

$$\mathrm{Vol}(S, g_{u,\mu}) = t^{m+1}V$$
$$\mathrm{diam}(S, g_{u,\mu}) \leq \pi$$

where V *is the volume of* S *with respect to* $(d\eta)^m \wedge \eta$*.*

Proof. Since $\mu = t^{-1}$, lemma 3.3 yields,

$$\begin{aligned}\mathrm{Vol}(S, g_{u,\mu}) &= \int_S (d\eta_{u,\mu})^m \wedge \eta_{u,\mu} = \mu^{-(m+1)} \int_S (d\eta_u)^m \wedge \eta_u \\ &= t^{m+1} \int_S (d\eta)^m \wedge \eta \\ &= t^{m+1}V.\end{aligned}$$

By theorem 2.4, we have

$$\mathrm{Ric}_{u,\mu}(X, Y) = \mathrm{Ric}^T_{u,\mu}(X, Y) - 2g_{u,\mu}(X, Y) \qquad \forall X, Y \in \mathrm{Ker}\,\eta_{u,\mu},$$

and by definition of $\eta_{u,\mu}$,

$$\mu g_{u,\mu}(X, Y) = g_u(X, Y) \qquad \forall X, Y \in \mathrm{Ker}\,\eta_{u,\mu}.$$

Since the transversal Ricci curvature is invariant under the multiplication by positive constant of a transversal metric, thus $\mathrm{Ric}^T_{u,\mu} = \mathrm{Ric}^T_u$, for all $X, Y \in \mathrm{Ker}\,\eta_{u,\mu}$. Then we have

$$\begin{aligned}\mathrm{Ric}^T_{u,\mu}(X,Y) &= \mathrm{Ric}^T_u(X,Y)\\ &\geq t(2m+2)g^T_u(X,Y)\\ &= (2m+2)t\mu g^T_{u,\mu}(X,Y)\\ &= (2m+2)t\mu g_{u,\mu}(X,Y).\end{aligned}$$

Therefore we have

$$\mathrm{Ric}_{u,\mu}(X,Y) \geq (2m+2)t\mu g_{u,\mu}(X,Y) - 2g_{u,\mu}(X,Y) \qquad \forall X, Y \in \mathrm{Ker}\,\eta_{u,\mu}.$$

Since we set

$$\mu = t^{-1}.$$

we have

$$\mathrm{Ric}_{u,\mu}(X,Y) \geq 2m g_{u,\mu}(X,Y) \qquad \forall X, Y \in \mathrm{Ker}\,\eta_{u,\mu}.$$

It follows from theorem 2.4 that

$$\begin{aligned}\mathrm{Ric}_{u,\mu}(X,\xi_\mu) =&2m\,\eta_{u,\mu}(X)\\ =&2m\,g_{u,\mu}(X,\xi_\mu), \qquad \forall X \in TS\end{aligned}$$

Therefore we obtain

$$\mathrm{Ric}_{u,\mu} \geq 2m g_{u,\mu} \geq (m-1)g_{u,\mu}.$$

By the Myers theorem, we have

$$\mathrm{diam}(S, g_{u,\mu}) \leq \pi.$$

□

Lemma 5.7. *Let $\triangle$ be a compact Sasakian manifold (S, g, η, ξ) and $\triangle_g$ the Laplacian with respect to g on S. The transversal Kähler metric $d\eta$ on S gives the basic Laplacian $\triangle^B_{d\eta}$ and the basic complex Laplacian $\Box^B_{d\eta}$ on S. Then we have*

$$\triangle_g u = \triangle^B_{d\eta} u = 2\Box^B_{d\eta} u,$$

for every basic function u on S.

Proof. As in Kähler geometry, we have $\triangle^B_{d\eta} u = 2\Box^B_{d\eta} u$. There is a relation between the Hodge star operator $*$ and the basic Hodge star operator $*_B$,

$$*\,(\eta \wedge \alpha) = *_B \alpha, \tag{13}$$

$$*\,\alpha = (-1)^p \eta \wedge *_B \alpha \tag{14}$$

for a basic p-form α. Since $d\eta$ is a basic 2-form, we have $d\eta \wedge *_B du = 0$. Then we obtain

$$\begin{aligned} \triangle_g u &= - * d * du = - * d(-\eta \wedge *_B du) && (15)\\ &= - *(\eta \wedge d *_B du) && (16)\\ &= = - *_B d_B *_B d_B u = \triangle^B u && (17) \end{aligned}$$

□

We can estimate u with the functional I.

Lemma 5.8. *Let (S, g, η, ξ) be a compact Sasakian manifold and u_t $(0 < t \leq 1)$ a family of basic functions with transversal Kähler form $d\eta_{u_t}$. We assume $\mathrm{Ric}^T(d\eta_{u_t}) \geq (2m+2)t d\eta_{u_t}$. Let G_g be the Green function with respect to g with a lower bound $\inf G_g \geq -K$. Then there exists a constant C which doesn't depend on t such that*

$$\begin{aligned} \operatorname*{osc}_S u_t &= \sup_S u_t - \inf_S u_t \\ &\leq I(0, u_t) + 2m\left(\frac{KV}{m!} + \frac{C}{t}\right) \end{aligned}$$

where V is the volume of S with respect to $(d\eta)^m \wedge \eta$.

Proof. Since $d\eta_{u_t} = d\eta + \sqrt{-1}\partial\bar{\partial} u_t$ is a transversal Kähler form, we have

$$\Box^B_{d\eta} u_t = \mathrm{tr}_{d\eta}(d\eta - d\eta_{u_t}) \leq m.$$

Then applying the fact 5.5 and lemma 5.7, we have an upper bound of u_t

$$\begin{aligned} u_t(x) &= \frac{1}{V}\int_S u_t (d\eta)^m \wedge \eta + \int_S \left(G_g(x,y) + K\right)(\triangle_g u_t)\frac{(d\eta)^m \wedge \eta}{m!} && (18)\\ &= \frac{1}{V}\int_S u_t (d\eta)^m \wedge \eta + \int_S \left(G_g(x,y) + K\right)(2\Box^B_{d\eta} u_t)\frac{(d\eta)^m \wedge \eta}{m!} && (19)\\ &\leq \frac{1}{V}\int_S u_t (d\eta)^m \wedge \eta + 2mK\frac{V}{m!}. && (20) \end{aligned}$$

Let $\triangle_{t,\mu}$ be the Laplacian and $G_{t,\mu}$ the Green function with respect to the Sasakian metric $g_{u_t,\mu}$ defined by (10). By proposition 5.6 and fact 5.5, we have

$$G_{t,\mu} \geq -\gamma(m,\varepsilon)\frac{\mathrm{diam}(S, g_{u_t,\mu})^2}{\mathrm{Vol}(S, g_{u_t,\mu})} \geq -\gamma(m,0)\frac{\pi^2}{t^{m+1}V}, \tag{21}$$

where $\mu = t^{-1}$ as in proposition 5.6. We denote by $2\Box^B_{t,\mu}$ the basic complex Laplacian with respect to the transversal Kähler form $d\eta_{u_t,\mu}$. Then it follows from lemma 5.7 that $\triangle_{t,\mu} u_t = 2\Box^B_{t,\mu} u_t$. By $d\eta_{u_t,\mu} = \mu^{-1}\left(d\eta + \sqrt{-1}\partial\bar{\partial} u_t\right)$, we have

$$\Box^B_{t,\mu} u_t = \mu \Box^B_{t,\mu} \mu^{-1} u_t = \mu\, \mathrm{tr}_{d\eta_{u_t,\mu}}(d\eta_\mu - d\eta_{u_t,\mu}) \geq -mt^{-1} \tag{22}$$

where $\eta_\mu = \mu^{-1}\eta$ and $\mu = t^{-1}$. By applying the fact 5.5 to $(S, g_{u_t,\mu})$, we have

$$\begin{aligned} u_t(x) =& \frac{1}{t^{m+1}V}\int_S u_t(d\eta_{u_t,\mu})^m \wedge \eta_{u_t,\mu} && (23)\\ &+ \int_S \left(G_{t,\mu}(x,y) + \gamma(m,0)\frac{\pi^2}{t^{m+1}V}\right)(\triangle_{t,\mu}u_t)\frac{(d\eta_{u_t,\mu})^m \wedge \eta_{u_t,\mu}}{m!} && (24)\end{aligned}$$

By (12), the first term (23) is given by

$$\frac{1}{t^{m+1}V}\int_S u_t(d\eta_{u_t,\mu})^m \wedge \eta_{u_t,\mu} = \frac{1}{V}\int_S u_t(d\eta_{u_t})^m \wedge \eta$$

By using (21) and (22), we have an estimate of (24),

$$\begin{aligned} &\int_S \left(G_{t,\mu}(x,y) + \gamma(m,0)\frac{\pi^2}{t^{m+1}V}\right)(\triangle_{t,\mu}u_t)\frac{(d\eta_{u_t,\mu})^m \wedge \eta_{u_t,\mu}}{m!}\\ =&\int_S \left(G_{t,\mu}(x,y) + \gamma(m,0)\frac{\pi^2}{t^{m+1}V}\right)(2\Box^B_{t,\mu}u_t)\frac{(d\eta_{u_t,\mu})^m \wedge \eta_{u_t,\mu}}{m!}\\ \\ \geq& -\frac{2m}{t}\gamma(m,\varepsilon)\frac{\pi^2}{t^{m+1}V}\frac{t^{m+1}V}{m!}\\ =& -2m\gamma(m,0)\frac{\pi^2}{t(m!)}\end{aligned}$$

Thus we obtain

$$u_t(x) \geq \frac{1}{V}\int_S u_t(d\eta_{u_t})^m \wedge \eta - 2m\gamma(m,0)\frac{\pi^2}{t(m!)} \tag{25}$$

From the upper bound (20) and the lower bound (25) we have the desired estimate

$$\begin{aligned}\operatorname*{osc}_S u_t =& \sup_S u_t - \inf_S u_t\\ \leq& I(0,u_t) + 2m\left(\frac{KV}{m!} + \frac{C}{t}\right).\end{aligned}$$

□

Lemma 5.9. *Let (S, g) be a compact Sasakian manifold. If we have a constant A such that*

$$I(0, u_t) \leq A$$

for a solution u_t of (9), then we have a constant B which doesn't depend on t such that

$$\|u_t\|_{C^{2,\varepsilon}} \leq B.$$

Proof. We denote by C_i a constant which does not depend on t.

By lemma 5.8, we have

$$t \operatorname*{osc}_S u_t \le t \left(I(0,u_t) + 4m \left(\frac{KV}{m!} + \frac{C}{t} \right) \right) \le C_1.$$

When we integrate (9) on S, we have

$$\begin{aligned} &\int_S \exp\left(-t(2m+2)u_t - (2m+2)L(0,u_t) + h\right) (d\eta)^m \wedge \eta \\ =&\int_S \frac{(d\eta + \sqrt{-1}\partial_B \bar{\partial}_B u_t)^m}{(d\eta)^m} (d\eta)^m \wedge \eta \\ =&\int_S (d\eta + \sqrt{-1}\partial_B \bar{\partial}_B u_t)^m \wedge \eta \\ =&\int_S (d\eta)^m \wedge \eta. \end{aligned}$$

Therefore there exists a point x_t such that

$$-t(2m+2)u_t(x_t) - (2m+2)L(0,u_t) + h(x_t) = 0. \tag{26}$$

Then for all $x \in S$, we have

$$\begin{aligned} &|-t(2m+2)u_t(x) - (2m+2)L(0,u_t) + h(x)| \\ =&|t(2m+2)u_t(x_t) - t(2m+2)u_t(x) - h(x_t) + h(x)| \\ \le& t \operatorname*{osc}_S u_t + 2 \sup_S |h| \le C_2. \end{aligned}$$

Hence we have

$$\begin{aligned} &\sup_S \left| \log \frac{(d\eta + \sqrt{-1}\partial_B \bar{\partial}_B u)^m}{(d\eta)^m} \right| \\ =&\sup_S |-t(2m+2)u_t - (2m+2)L(0,u_t) + h| \\ \le& C_2. \end{aligned}$$

Thus, by Yau [12], we have

$$\operatorname*{osc}_S u_t \le C_3.$$

as in compact Kähler manifolds.

Next, we shall estimate $\sup_S |u_t|$. We use a path su_t to compute the functional L for $0 \le s \le 1$. Then we have

$$\begin{aligned} &|L(0,u_t) - u_t(x_t)| \\ =&\left| \frac{1}{V} \int_0^1 \left(\int_S (u_t - u_t(x_t))(d\eta + s\sqrt{-1}\partial_B \bar{\partial}_B u_t)^m \wedge \eta \right) ds \right| \\ \le& \frac{1}{V} \int_0^1 \left(\int_S \operatorname*{osc}_S u_t (d\eta + s\sqrt{-1}\partial_B \bar{\partial}_B u_t)^m \wedge \eta \right) ds \\ =&\operatorname*{osc}_S u_t \\ \le& C_3. \end{aligned}$$

By (26), we have

$$\begin{aligned}(2m+2)(1+t)|u_t(x_t)| =&|(2m+2)u_t(x_t) - (2m+2)L(0,u_t) + h(x_t)|\\ =&(2m+2)|u_t(x_t) - L(0,u_t)| + |h(x_t)|\\ \leq& C_4.\end{aligned}$$

Thus we have $|u_t(x_t)| \leq C_5$. Hence for all $x \in S$, we have

$$\begin{aligned}|u_t(x)| =&|u_t(x) - u_t(x_t) + u_t(x_t)|\\ \leq& \operatorname*{osc}_S u_t + |u_t(x_t)|\\ \leq& C_6.\end{aligned}$$

Therefore we obtain

$$\|u_t\|_{C^{2,\varepsilon}} \leq C_7.$$

□

6 Proof of main theorem

First we prove the next proposition.

Theorem 6.1. *Let (S,g) be a compact Sasakian manifold and $\tau \in (0,1)$.*

(i) *For $t \in [0,\tau]$, if there exists smooth one-parameter families u_t, u'_t of solutions of (9) , then $u_t = u'_t$.*

(ii) *If there exists a solution u_τ of (9) at $t = \tau$, u_τ uniquely extends to a smooth family $\{u_t \mid 0 \leq t \leq \tau\}$ of solutions of (9).*

(iii) *If there exist two solutions u_τ, u'_τ of (9) at $t = \tau$, then $u_\tau = u'_\tau$.*

Proof. (i) By proof of proposition 5.3, the solution of (9) is locally unique. Therefore we shall show that u_0 is unique. Since u_0 is a solution of (9) we have

$$\int_S (d\eta)^m \wedge \eta = \int_S (d\eta_{u_0})^m \wedge \eta = \exp(-(2m+2)L(0,u_0)) \int_S e^h (d\eta)^m \wedge \eta.$$

By definition of h, we have

$$\int_S (e^h - 1)(d\eta)^m \wedge \eta = 0$$

Therefore we obtain

$$L(0,u_0) = 0.$$

Then there exist a solution which is unique up to an additive constant by [12] and [5]. The condition $L(0,u) = 0$ says that there is no difference by the additive constant. Therefore u_0 is unique and u_t is unique also.

(ii) Let $I_2 = \{t \in [0,1] \mid \text{the equation (9) has solutions for } t\}$. Since we already showed that I_2 is open, it suffices to show that I_2 is closed. By proposition 4.2 and lemma 5.4, we have

$$\begin{aligned} I(0,u_t) \leq& (m+1)(I(0,u_t) - J(0,u_t)) \\ \leq& (m+1)(I(0,u_\tau) - J(0,u_\tau)). \end{aligned}$$

The right hand side is independent of t. Therefore, by lemm 5.9, we have

$$\|u_t\|_{C^{2,\varepsilon}} \leq C.$$

Hence it follows from the Ascoli-Arzelà theorem that I_2 is closed.

(iii) From 2, u_τ, u'_τ are extended in u_t, u'_t $(t \in [0,\tau])$. From 1 we have $u_t = u'_t$. □

We are in a position to prove main theorem.

Theorem 1.1. *Let (S, ξ, η, Φ) be a compact Sasakian manifold. We assume that S doesn't admit nontrivial Hamiltonian holomorphic vector fields. If S has a Sasaki-Einstein metric, then the Sasaki-Einstein metric is unique. In other words, if there are two Sasaki-Einstein metrics ω_1 and ω_2 on S, then $\omega_1 = \omega_2$.*

Proof. If we have a solution of (8), then we have a solution of (9). Therefore theorem 6.1 is true for (8). In particular, by proposition 5.3, if S doesn't have nontrivial Hamiltonian holomorphic vector fields, then I_1 is open in $t = 1$. Therefore if there are two Sasaki-Einstein metrics ω_1 and ω_2, then $\omega_1 = \omega_2$. □

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